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GLOBAL AND LOCAL ADVERTISING STRATEGIES: A DYNAMIC MULTI-MARKET OPTIMAL CONTROL MODEL

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Abstract. Differential games have been widely used to model advertising strategies of companies. Nevertheless, most of these studies have concentrated on the dynamics and market structure of the problem, neglecting their multi-market dimension. Since nowadays competition typically operates on multi-product contexts and usually in geographically separated markets, the optimal advertising strategies must take into consideration the different levels of disaggregation, especially, for example, in retail multi-product and multi-store competition contexts. In this paper, we look into the decision-making process of a multi-market company that has to decide where, when and how much money to invest in advertising. For this purpose, we develop a model that keeps the dynamic and oligopolistic nature of the traditional advertising game introducing the multi-market dimension of today's economies, while differentiating global (i.e. national TV) from local advertising strategies (i.e. a price discount promotion in a particular store). It is important to note, however, that even though this problem is real for most multi-market companies, it has not been addressed in the differential games literature. On the more technical side, we steer away from the traditional aggregated dynamics of advertising games in two aspects. Firstly, we can model different markets at once, obtaining a global instead of a local optimum, and secondly, since we are incorporating a variable that is common to markets, the resulting equations systems for every market are now coupled. In other words, one's decision in one market does not only affect one's competition in that particular market; it also affects one's decisions and one's competitors in all markets.

1. Introduction.

The dynamic modeling of advertising competition has a long-standing tradition in operations research and economics starting with the seminal works of Vidale and Wolfe [32] and Kimball [22], who developed the Vidale-Wolfe model and the Lanchester model[1], respectively.

The Vidale-Wolfe model, on the one hand, proposes a direct relationship between the rate of change of the advertising efforts of companies and their sales, while the Lanchester model, on the other hand, aims at the market share evolution of two competitors given their advertising level just as it would occur in a military combat

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[1] Strictly speaking the Lanchester equations were developed before to model the dynamic of military combat, see for example, [17] for a recent application in that direction.

model. Moreover, applying a more economic view of the problem, Nerlove and Arrow [27] developed a model where advertising is an instrument to increase the stock of goodwill or reputation, including the effects of advertising expenses incurred by a company on the demand for its goods. Over time, all these approaches have converged onto a game-theoretic view of the problem that has attracted a great deal of research in the last thirty years, particularly in the subject of differential games. The differential game approach aims at obtaining a dynamic Nash equilibrium in advertising strategies. For excellent surveys of the subject, see [20, 14, 9, 12, 21, 18, 30].

Most early research on differential games and advertising competition considered duopolies and optimal open-loop equilibrium solutions. However, researchers developed more complete and sophisticated models in recent years; for instance, see [8, 16, 15, 4, 21, 13, 28, 23]. Indeed, in order to get a quantitative idea of when and how much to invest in advertising, it seems necessary to recognize that advertising games are dynamic and oligopolistic in nature, as both literature and practice suggest. For empirical analyses see [11, 6, 5, 16, 2].

Nevertheless, most research has concentrated on the dynamics and market structure of the problem, neglecting its multi-market dimension[2]. Nowadays, companies typically compete on geographically separated and/or segmented markets; consequently, the multi-market modeling of advertising strategies is a crucial issue.

Thus, supermarket chains, multinationals, franchises as well as food, gasoline, supplies and service retailers (i.e. banks and telecommunications, to mention a couple), compete for market shares in different locations within a city, region or country. Their results, however, differ since they usually segment their strategies by different types of consumers, specific places, formats (distribution channels) and advertising. By way of example, see [7, 1, 10, 19].

Furthermore, different markets may involve clients with different social backgrounds and preferences, markets with different levels of revenue and competition, different rates of sales growth, and different costs and channels of advertising. Within this context, we may consider the multi-market problem as a type of market segmentation. To that effect, you may see ref. [3].

Within this context of heterogeneity, a general model considering average parameters could give sub-optimal results. On the other hand, locally optimal strategies will always be sub-optimal strategies, unless there is a perfect homogeneity between the store's characteristics and environments, in terms of the effectiveness of their strategy, costs and levels of competitiveness.

Besides, on the practical side of the problem, retail chains have to choose whether to undertake global (i.e. worldwide, national or regional) or local (i.e. a city zone or region, or even an entire country) advertising efforts or an optimal combination of both; again, applying different spatial levels of effectiveness and costs to both advertising strategies. This should take into consideration that global advertising campaigns usually substitute local efforts somehow.

Let us take, by way of example, two food retailers: W and T compete in almost every corner of a country and they are currently expanding their operations worldwide. In a high-income location, W could choose to have a higher market share due to the level of monetary margin obtained. To that effect, W implements additional promotions and starts installing huge ads on the streets as part of a very aggressive

[2]An exception is [26], who proposed a partial differential equation (PDE) to characterize the goodwill dynamics of a monopoly in both the space and time dimensions.

advertising campaign. At the same time, at a national level, W could decide to decrease its expense, cutting down television spots and its support to the national football team. T will have to respond to W at the global level and in the specific market of interest. Current modeling strategies cannot quantitatively answer the question of W's optimal level of global advertising, nor the optimal response of T given W's strategy.

Given this heterogeneous environment, companies usually have to decide as to when and how much to invest in advertising, but also as to which market to allocate their advertising funds. Moreover, retail chains have to choose whether to undertake global (i.e. worldwide, national or regional) or local (i.e. a city zone or region, or even an entire country) advertising efforts or an optimal combination of both.

The present investigation specifically intends to addresses this gap in the literature, presenting a dynamic spatial oligopoly model. Specifically, we propose a model that, keeping the dynamic and oligopolistic characteristic of the advertising game, introduces the multi-market dimension of multi-store companies, now considering global and local advertising strategies. Given the complexity of this problem, an analytical solution is very difficult or impossible to obtain. Thus, we developed a numerical model to simulate a centralized versus a decentralized decision-making process for different advertising strategies. Our results highlight the importance of multi-market modeling, depending on the effectiveness of global advertising and market heterogeneity.

The structure of this paper is the following: in section 2 we give a very short review of the different models associated to the advertising problem. In Section 3, we formulate the dynamic spatial advertising problem in a very general fashion, considering n companies and K markets in an oligopolistic setting. Section 4 shows the particular case of two players and two specific markets. In this section, we seek for a numerical solution applicable to several cases with the purpose of analyzing the importance of global advertising strategies, the asymmetries of the players and the geographical presence of the companies. Finally, we provide a detailed summary and suggest guidelines for future research.

2. **The basic dynamic advertising model.** The Lanchester model is a mathematical representation of the dynamics between two opposite forces fighting to annihilate each other [24]. As mentioned above, several authors saw this as an analogy between armies and companies fighting for market share while using their respective advertising budgets as their main weapons and modifying the basic and famous Lanchester equations. To that effect, you may refer to [22] for the first application. We describe a duopolistic model of dynamic advertising competition as follows:

$$\begin{aligned} \dot{x}_1 &= \sigma_1\mu_1(1-x_1) - \sigma_2\mu_2 x_1 \\ \dot{x}_2 &= \sigma_2\mu_2(1-x_2) - \sigma_1\mu_1 x_2 \end{aligned} \tag{1}$$

In simple words, the dynamics that govern the market share of companies (understood as a company's sales percentage divided by total market sales) depend positively from the advertising expense (budgetary effort) of the μ_1 company and its corresponding σ_1 effectiveness, and negatively, on the advertising expense (budgetary effort) of the rival μ_2 company and its corresponding σ_2 effectiveness. Please refer to [32] and [25]. Years later, [29, 31]introduced an important extension to this model:

$$\begin{aligned} \dot{x}_1 &= \sigma_1\mu_1\sqrt{1-x_1} - \sigma_2\mu_2\sqrt{x}_1 \\ \dot{x}_2 &= \sigma_2\mu_2\sqrt{1-x_2} - \sigma_1\mu_1\sqrt{x_2} \end{aligned} \tag{2}$$

The main feature of this extended formulation was to introduce diminishing marginal returns on advertising and, consequently, on its real-world application as word-of-mouth and excess advertising effects. The work of Chintagunta and Jain [5] complements this potentially better fit to the real world of applications.

The problem is completed as an optimal control problem adding the objective function of the firm:

$$\underset{\mu_1}{max} \quad J_1 = \int_0^T e^{-r_1 t}\left[q_1 \cdot x_1 - \tfrac{1}{2}c_1\mu_1^2\right]dt \tag{3}$$

subject to

$$\dot{x}_1 = \sigma_1\mu_1\sqrt{1-x_1} - \sigma_2\mu_2\sqrt{x}_1 \tag{4}$$

where the company maximizes dynamically its profits J_1 (i.e. incomes minus costs) for the next T periods using a discount rate of r_1. Its income is obtained by multiplying its market share (x_1) by its gross profit per unit of market share (q_1), while its costs are assumed as a quadratic function of the advertising effort; see for example [31, 9, 21]. Company 1 wishes to choose its efforts in advertising μ_1, which depends upon the same decision of its rival. For the optimal open-loop solution of the differential game shown in 3, we define each company's Hamiltonian current-value, as follows[3]:

$$H^1(x_1,\mu_1,\lambda_1,t) = q_1 \cdot x_1 - \frac{1}{2}c_1\mu_1^2 + \lambda_1\left[\sigma_1\mu_1\sqrt{1-x_1} - \sigma_2\mu_2\sqrt{x}_1\right]$$

The control of this problem, the Nash Equilibrium, is then defined as:

$$\mu_1(t) = \frac{\lambda_1}{c_1}\sqrt{1-x_1} \tag{5}$$

In order to solve 5, we need to solve a system of (coupled) differential equations of each company's co-state equation, as follows[4]:

$$\dot{\lambda}_1 = \frac{\sigma_1^2\lambda_1^2}{2c_1} + \left[\frac{\sigma_2^2\lambda_2}{2c_1} + r_1\right]\lambda_1 - q_1 \tag{6}$$

Equation 6 shows that $\dot{\lambda}_1$ does not depend on the state variables x_1, x_2, thus, for the stationary trajectories when $\dot{\lambda}_1 = 0$, the phase space of the system(λ, x) is represented by lines parallel to the x axis. On the other hand, the unique solution of the state equation is:

$$\dot{x}_1(t) = \frac{\sigma_1^2\eta_1(t)}{c_1}(1-x_1(t)) - \frac{\sigma_2^2\eta_2(t)}{c_2}x_1(t) \tag{7}$$

where $\eta_1(t)$ and $\eta_2(t)$ are the solutions of 6. Finally, the solutions of 6 and 7 are replaced into 5 in order to find the solution to the optimal advertising problem. Recently, [13] found the equilibrium for Nash's feedback equilibrium on advertising strategies for an oligopoly model. Most of the current research in the area modifies and extends the model explained above including technical generalizations and/or highlighting new real problems in the industry. The objective of our research is to advance in these two aspects: on the one hand, to solve a heretofore-unsolved

[3]To save space, we will only show the equations corresponding to firm1.

[4]Considering the following normalization of the variables:$\lambda_1 \to \frac{\lambda_1}{e^{-r_1 t}}$ and $\lambda_2 \to \frac{\lambda_2}{e^{-r_2 t}}$

technical problem and, on the other, to tackle a real and contemporaneous industrial problem in this area.

Firstly, we must recognize that most companies today are essentially multi-market. This means that they face several problems of optimal advertising strategies as described above, but all of them with potentially different parameters. For example, the market share of a supermarket chain in one part of the city could be very different in another location. Secondly, we also introduce to the model the possibility of doing local advertising and apply global advertising strategies. The global advertising strategy could be, for example, to advertise using TV spots that constitute an effort for every different geographical market. This problem is very real today in many industries; however, to date the literature does not offer a control problem toward its analysis. Our model is, therefore, the first to deal directly with this question of how to distribute a company's advertising efforts between local and global strategies. Indeed, global advertising strategies seem to be very appealing because they reach all markets at the same time; nevertheless, they are usually quite expensive. Hence, there is no intuitive answer to this question and, within this context, our model could be very useful, especially considering the great amount of money that multi-market companies spend yearly on advertising.

On the more technical side, we depart from the above-indicated game in two aspects. One is that we can model different markets at once obtaining a global optimum and, the other, is that since we are considering a variable that is common to all the different markets, all the resulting systems for every market are indeed coupled. In other words, one's decision in one's market does not only affect one's competition in that particular market, but it also affects one's decisions in all markets, and, likewise, for my competitors. Thus, the problem gets exponentially more complex, requiring a general Nash Equilibrium solution. Again, to the best of our knowledge, no problem like this (within the context of the management science) has yet been thus proposed and solved numerically.

3. **The general model.** Consider a dynamic oligopoly with n firms. We use the index $i = 1, ..., n$ to represent the participant firms and the index $k = 1, ..., K$ to set the number of markets. We begin by listing the main notation:

J_i	Profit function of player i.
$x_{ik}(t)$	Market share of player i at location k.
q_{ik}	Gross profit rate per unit of market share of player i at location k.
Q_{ik}	Second order gross profit rate per unit of market share of player i at location k.
b_{ik}	Linear local advertising cost of player i at location k.
B_{ik}	Second order local advertising cost of player i at location k.
e_i	Linear global advertising cost of player i at location k.
E_i	Second order global advertising cost of player i.
σ_{ik}	Effectiveness of local advertising of player i at location k.
σ_i	Effectiveness of global advertising of player i at location k.
r_i	Discount rate of player i.
$\mu_{ik}(t)$	Local advertising effort of player i at location k.
$\mu_i(t)$	Global advertising effort of player i.

Table 1. Notation

Each firm maximize its aggregate profits, gaining market share from its rivals through its advertising, and also loses market share to its rivals due to their advertising in each particular market. In this analysis, the location is explicitly modeled, differentiating from global (worldwide, national or regional) or local (a zone in a city or a region, or even a country) advertising efforts. The results of the model are the optimal allocation of global and local advertising, considering the strategy of its rival.

$x_{ik}(t)$ denotes firm i's market share at time t and location k, assuming that the size of the total market is constant over time. Normalizing the total market to 1, the rest of the market share becomes $1 - x_{ik}(t)$. Firm i wishes to choose its advertising efforts; $\mu_{ik}(t)$ and $\mu_i(t)$, at local and global levels respectively, $\forall\, t \,\epsilon\, [0, T]$ and $k = 1, ..., K$ such that its payoffs is maximized subject to $\mu_{ik}(t) \geq 0$ and $\mu_i(t) \geq 0 \;\forall\, t \,\epsilon\, [0, T]$ and $k = 1, ..., K$. The state space is specified by the constraints $0 \leq x_{ik}(t) \leq 1$, $\forall\, t \,\epsilon\, [0, T]$ and $k = 1, ..., K$.

This dynamic multi-market differential game is defined by the n competitors functionals J_i $(i = 1..n)$ given by:

$$J_i = \int_0^T e^{-r_i t} \left[\sum_{k=1}^{K} \Big(q_{ik}\, x_{ik}(t) + \frac{1}{2} Q_{ik}\; x_{ik}(t)^2 - b_{ik}\, \mu_{ik}(t) - \frac{1}{2} B_{ik} \mu_{ik}(t)^2 \Big) - e_i \mu_i(t) - \frac{1}{2} E_i \mu_i(t)^2 \right] dt \tag{8}$$

subject to

$$\frac{dx_{ik}}{dt} = \alpha_{ik}(t) \sqrt{1 - x_{ik}(t)} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \alpha_{jk}(t) \sqrt{1 - x_{jk}(t)} \qquad \begin{array}{l} i = 1, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \tag{9}$$

where the coefficients $\alpha_{ik}(t)$ depend on the control variables according to

$$\alpha_{ik}(t) = \sigma_{ik}\; \mu_{ik}(t) + \sigma_i\; \mu_i(t), \qquad \begin{array}{l} i = 1, \ldots, n. \\ k = 1, \ldots, K. \end{array} \tag{10}$$

Note that for fixed k, there are $(n-1)$ dynamical equations for the $x_{ik}(t)$. It is due to the restrictions

$$\sum_{j=1}^{n} x_{jk}(t) = 1 \qquad k = 1,\ ...,\ K \tag{11}$$

Thus, in the set of n state variables $\{x_{jk}(t)\}|_{j=1,\ 2\ ,...,\ n}$, there are only $(n-1)$ independent ones. It is assumed here the set $\{x_{1k},\ x_{2k},\ ..,\ x_{n-1k}\}$ is the independent state variables set (for fixed k) and that the last variable x_{nk} is the dependent one, and given by

$$x_{nk} = 1 - x_{1k} - x_{2k} - x_{3k} - ... - x_{n-1k} = 1 - \sum_{j=1}^{n-1} x_{jk} \tag{12}$$

Equation (9) can written in terms of the independent state variables as

$$\frac{dx_{ik}}{dt} = \sum_{j=1}^{n-1}\; \gamma_{ij}\;\, \alpha_{jk}(t) \sqrt{1 - x_{jk}(t)} - \alpha_{nk}(t) \sqrt{\sum_{l=1}^{n-1} x_{lk}(t)} \qquad \begin{array}{l} i = 1, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \tag{13}$$

where

$$\gamma_{ij} = \begin{cases} +1 & \text{if } i = j \\ -1 & \text{if } i \neq j \end{cases} \tag{14}$$

In order to get a solution, we first construct the Hamiltonian for each firm. Written in terms of the independent state variables, the Hamiltonian for each of the first $(n-1)$ players is

$$\begin{aligned} H^i =& e^{-r_i t}\left[\sum_{k=1}^{K}\Big(q_{ik}\ x_{ik}(t) + \frac{1}{2}Q_{ik}\ x_{ik}(t)^2 - b_{ik}\ \mu_{ik}(t) - \frac{1}{2}B_{ik}\ \mu_{ik}(t)^2\Big)\right. \\ & \left. - e_i\ \mu_i(t) - \frac{1}{2}E_i\ \mu_i(t)^2\right] + \sum_{m=1}^{n-1}\sum_{k=1}^{K}\lambda^i_{mk}(t)\left[\sum_{j=1}^{n-1}\gamma_{mj}\ \alpha_{jk}(t)\sqrt{1-x_{jk}(t)}\right. \\ & \left. - \alpha_{nk}(t)\sqrt{\sum_{l=1}^{n-1}x_{lk}(t)}\right], \quad i = 1,\dots,(n-1), \end{aligned} \tag{15}$$

and for the last player $(i = n)$

$$\begin{aligned} H^n =& e^{-r_n t}\Big[\sum_{k=1}^{K}\Big(q_{nk}\ (1-\sum_{j=1}^{n-1}x_{jk}) + \frac{1}{2}Q_{nk}\ (1-\sum_{j=1}^{n-1}x_{jk})^2 \\ & - b_{nk}\ \mu_{nk}(t) - \frac{1}{2}B_{nk}\ \mu_{nk}(t)^2\Big) - e_n\ \mu_n(t) - \frac{1}{2}E_n\ \mu_n(t)^2\Big] \\ & + \sum_{m=1}^{n-1}\sum_{k=1}^{K}\lambda^n_{mk}(t)\ \left[\sum_{j=1}^{n-1}\ \gamma_{mj}\ \alpha_{jk}(t)\sqrt{1-x_{jk}(t)} - \alpha_{nk}(t)\sqrt{\sum_{l=1}^{n-1}x_{lk}(t)}\right] \end{aligned} \tag{16}$$

Following Potryagin's Maximum Principle, firstly, the optimization of the control variables gives

$$\frac{\partial H^i}{\partial \mu_{ik}} = 0, \quad \frac{\partial H^i}{\partial \mu_i} = 0, \quad \frac{\partial H^n}{\partial \mu_{nk}} = 0, \quad \frac{\partial H^n}{\partial \mu_n} = 0; \qquad i = 1,..,(n-1) \tag{17}$$

which have to be solved to find the optimal controls

$$\begin{aligned} &\mu_{ik}(t)\ ,\ \mu_i(t); \qquad i = 1,\ ..,\ (n-1); \quad k = 1,\ ..,\ K. \\ &\mu_{nk}(t)\ ,\ \mu_n(t); \qquad k = 1,\ ..,\ K. \end{aligned} \tag{18}$$

In fact, the optimal controls variables (17) are:

$$\mu_{ik}(t) = \frac{\sigma_{ik}\sqrt{1-x_{ik}(t)}\left(\sum_{m=1}^{n-1}\gamma_{mi}\ \lambda^i_{mk}(t)\right)}{e^{-r_i t}\ B_{ik}} - \frac{b_{ik}}{B_{ik}}, \qquad \begin{matrix} i = 1,\dots,(n-1). \\ k = 1,\dots,K. \end{matrix} \tag{19}$$

$$\mu_i(t) = \frac{\sum_{k=1}^{K}\sigma_i\sqrt{1-x_{ik}(t)}\left(\sum_{m=1}^{n-1}\gamma_{mi}\lambda^i_{mk}(t)\right)}{e^{-r_i t}\ E_i} - \frac{e_i}{E_i} \qquad i = 1,\dots,(n-1). \tag{20}$$

$$\mu_{nk}(t) = -\frac{\sigma_{nk}\ \sqrt{\sum_{l=1}^{n-1} x_{lk}(t)}\Big(\sum_{m=1}^{n-1} \lambda_{mk}^{n}(t)\Big)}{e^{-r_n t}\ B_{nk}} - \frac{b_{nk}}{B_{nk}} \qquad k = 1, \ldots, K. \tag{21}$$

$$\mu_{n}(t) = -\frac{\sum_{k=1}^{K} \sigma_{n}\sqrt{\sum_{l=1}^{n-1} x_{lk}(t)}\Big(\sum_{m=1}^{n-1} \lambda_{mk}^{n}(t)\Big)}{e^{-r_n t}\ E_{n}} - \frac{e_{n}}{E_{n}} \tag{22}$$

Note that the net effect of the linear terms in μ_{ik} and μ_i associated with the b_{ik} and e_i parameters in the functionals (3), is to reduce in a rigid way the value of the optimal advertising efforts according to equations (19) to (22).

For optimal open loop strategies, the dynamical equations of the Lagrangian multipliers are

$$\dot{\lambda}_{jk}^{i} = -\frac{\partial H^{i}}{\partial x_{jk}}, \qquad \begin{array}{l} i = 1, \ldots, (n-1). \\ j = 1, 2, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \tag{23}$$

$$\dot{\lambda}_{jk}^{n} = -\frac{\partial H^{n}}{\partial x_{jk}}, \qquad \begin{array}{l} j = 1, 2, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \tag{24}$$

that is

$$\begin{aligned} \dot{\lambda}_{jk}^{i} = -e^{-r_i t}\delta_{ij}(q_{ik} + Q_{ik}\ x_{ik}) + \sum_{m=1}^{n-1} \lambda_{mk}^{i} \Bigg[\frac{\gamma_{mj}\ \Big(\sigma_{jk}\ \mu_{jk}(t) + \sigma_{j}\ \mu_{j}(t)\Big)}{2\sqrt{1 - x_{jk}}} \\ + \frac{\Big(\sigma_{nk}\ \mu_{nk}(t) + \sigma_{n}\ \mu_{n}(t)\Big)}{2\sqrt{\sum_{l=1}^{n-1} x_{lk}}} \Bigg], \qquad \begin{array}{l} i = 1, \ldots, (n-1). \\ j = 1, 2, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \dot{\lambda}_{jk}^{n} = e^{-r_n t} \Bigg[q_{nk} + Q_{nk}(1 - \sum_{p=1}^{n-1} x_{pk}) \Bigg] + \sum_{m=1}^{n-1} \lambda_{mk}^{n} \Bigg[\frac{\gamma_{mj}\ \Big(\sigma_{jk}\ \mu_{jk}(t)\sigma_{j}\ \mu_{j}(t)\Big)}{2\sqrt{1 - x_{jk}}} + \\ + \frac{\Big(\sigma_{nk}\ \mu_{nk}(t) + \sigma_{n}\ \mu_{n}(t)\Big)}{2\sqrt{\sum_{l=1}^{n-1} x_{lk}}} \Bigg], \qquad \begin{array}{l} j = 1, 2, \ldots, (n-1). \\ k = 1, \ldots, K. \end{array} \end{aligned} \tag{26}$$

In order to obtain a flavor of the above equations, the 2×2 case will be analyzed in detail in the following section.

4. **The 2 firms - 2 market case.** In this section, a model for the 2 firms - 2 market case is developed. Firstly, the equations for this particular case are derived from the general model. Secondly, the bang-bang control of the duopoly game is analyzed. Finally, a numerical solution is found for several cases of studies, with the aim of analyzing the importance of global advertising strategies, asymmetries of the players, and the geographical presence of firms.

4.1. **The basic 2x2 Model.** In this case the state variables $x_{1k}(t)$ and $x_{2k}(t)$ are restricted by $x_{1k}(t) + x_{2k}(t) = 1$ for each $k = 1,\ 2$ and for all t. Hence, there is only two independent state variables which is chosen to be x_{1k} $(k = 1,\ 2)$. The equation of motion for each of these dynamical variables is

$$\dot{x}_{1k}(t) = \alpha_{1k}(t)\sqrt{1 - x_{1k}(t)} - \alpha_{2k}(t)\sqrt{1 - x_{2k}(t)} \qquad k = 1,\ 2 \tag{27}$$

or in terms of the independent variable x_{1k}

$$\dot{x}_{1k}(t) = \alpha_{1k}(t)\sqrt{1 - x_{1k}(t)} - \alpha_{2k}(t)\sqrt{x_{1k}(t)} \qquad k = 1,\ 2 \tag{28}$$

where

$$\alpha_{1k}(t) = \sigma_{1k}\ \mu_{1k}(t) + \sigma_1\ \mu_1(t) \qquad k = 1,\ 2 \tag{29}$$

and

$$\alpha_{2k}(t) = \sigma_{2k}\ \mu_{2k}(t) + \sigma_2\ \mu_2(t) \qquad k = 1,\ 2 \tag{30}$$

The functional for the player 1 and 2 are

$$\begin{aligned} J_1 = \int_0^T e^{-r_1 t}\Bigg[\sum_{k=1}^{2}\Big(&q_{1k}\ x_{1k}(t) + \frac{1}{2}Q_{1k}\ x_{1k}(t)^2 - b_{1k}\ \mu_{1k}(t) \\ &- \frac{1}{2}B_{1k}\ \mu_{1k}(t)^2\Big) - e_1\ \mu_1(t) - \frac{1}{2}E_1\ \mu_1(t)^2\Bigg]\ dt \end{aligned} \tag{31}$$

and

$$\begin{aligned} J_2 = \int_0^T e^{-r_2 t}\Bigg[\sum_{k=1}^{2}\Big(&q_{2k}\ (1 - x_{1k}(t)) + \frac{1}{2}Q_{2k}\ (1 - x_{1k}(t))^2 - b_{2k}\ \mu_{2k}(t) \\ &- \frac{1}{2}B_{2k}\ \mu_{2k}(t)^2\Big) - e_2\ \mu_2(t) - \frac{1}{2}E_2\ \mu_2(t)^2\Bigg]\ dt \end{aligned} \tag{32}$$

subject to the equation (28). The corresponding Hamiltonians are

$$\begin{aligned} H^1 =& e^{-r_1 t}\Bigg[\sum_{k=1}^{2}\Big(q_{1k}\ x_{1k}(t) + \frac{1}{2}Q_{1k}\ x_{1k}(t)^2 - b_{1k}\ \mu_{1k}(t) - \frac{1}{2}B_{1k}\ \mu_{1k}(t)^2\Big) \\ &- e_1\ \mu_1(t) - \frac{1}{2}E_1\ \mu_1(t)^2\Bigg] \\ &+ \sum_{k'=1}^{2}\lambda^1_{1k'}\Big(\alpha_{1k'}(t)\sqrt{1 - x_{1k'}(t)} - \alpha_{2k'}(t)\sqrt{x_{1k'}(t)}\Big) \end{aligned} \tag{33}$$

and

$$\begin{aligned} H^2 =& e^{-r_2 t}\Bigg[\sum_{k=1}^{2}\Big(q_{2k}\ (1 - x_{1k}(t)) + \frac{1}{2}Q_{2k}\ (1 - x_{1k}(t))^2 - b_{2k}\ \mu_{2k}(t) \\ &- \frac{1}{2}B_{2k}\ \mu_{2k}(t)^2\Big) - e_2\ \mu_2(t) - \frac{1}{2}E_2\ \mu_2(t)^2\Bigg] \\ &+ \sum_{k'=1}^{2}\lambda^2_{1k'}\Big(\alpha_{1k'}(t)\sqrt{1 - x_{1k'}(t)} - \alpha_{2k'}(t)\sqrt{x_{1k'}(t)}\Big) \end{aligned} \tag{34}$$

So the Pontryagin equation for this case are for player 1

$$\dot{\lambda}^1_{1k} = -\frac{\partial H^1}{\partial x_{1k}}, \qquad \dot{x}_{1k} = \frac{\partial H^1}{\partial \lambda^1_{1k}}, \qquad \frac{\partial H^1}{\partial \mu_{1k}} = 0, \qquad \frac{\partial H^1}{\partial \mu_1} = 0. \qquad k = 1,\ 2$$

and for player 2

$$\dot{\lambda}^2_{1k} = -\frac{\partial H^2}{\partial x_{1k}}, \qquad \dot{x}_{1k} = \frac{\partial H^2}{\partial \lambda^2_{1k}}, \qquad \frac{\partial H^2}{\partial \mu_{2k}} = 0, \qquad \frac{\partial H^2}{\partial \mu_2} = 0. \qquad k = 1,\ 2$$

The equations for the control variables $\mu_{1k},\ k = 1,\ 2,\ \mu_{2k},\ k = 1,\ 2,\ \mu_1$ and μ_2 gives the following expression for the control variables in terms of the state variable x_{1k} and Lagrange multiplier $\lambda^1_{1k},\ k = 1,\ 2$ and $\lambda^2_{1k},\ k = 1,\ 2$ is for the player 1

$$\mu_{1k}(t) = \frac{\lambda^1_{1k}\ \sigma_{1k}\ \sqrt{1 - x_{1k}(t)}}{e^{-r_1 t}\ B_{1k}} - \frac{b_{1k}}{B_{1k}}, \qquad k = 1,\ 2 \tag{35}$$

$$\mu_1(t) = \frac{\sum_{k'} \sigma_1\ \lambda^1_{1k'}\ \sqrt{1 - x_{1k'}(t)}}{e^{-r_1 t}\ E_1} - \frac{e_1}{E_1} \tag{36}$$

and for the player 2 are

$$\mu_{2k}(t) = -\frac{\lambda^2_{1k}\ \sigma_{2k}\ \sqrt{x_{1k}(t)}}{e^{-r_2 t}\ B_{2k}} - \frac{b_{2k}}{B_{2k}}, \qquad k = 1,\ 2 \tag{37}$$

$$\mu_2(t) = -\frac{\sum_{k'} \sigma_2\ \lambda^2_{1k'}\ \sqrt{x_{1k'}(t)}}{e^{-r_2 t}\ E_2} - \frac{e_2}{E_2} \tag{38}$$

The dynamical equations for the Lagrangian multipliers give the following system of coupled differential equations

$$\dot{\lambda}^1_{1k} = -e^{-r_1 t}(q_{1k} + Q_{1k}\ x_{1k}) + \lambda^1_{1k}\left(\frac{\alpha_{1k}}{2\sqrt{1 - x_{1k}}} + \frac{\alpha_{2k}}{2\sqrt{x_{1k}}}\right), \qquad k = 1,\ 2 \tag{39}$$

and

$$\dot{\lambda}^2_{1k} = e^{-r_2 t}\big(q_{2k} + Q_{2k}(1 - x_{1k})\big) + \lambda^2_{1k}\left(\frac{\alpha_{1k}}{2\sqrt{1 - x_{1k}}} + \frac{\alpha_{2k}}{2\sqrt{x_{1k}}}\right), \qquad k = 1,\ 2 \tag{40}$$

Considering the following normalization of the variables:$\lambda_1 \to \frac{\lambda_1}{e^{-r_1 t}}$ and $\lambda_2 \to \frac{\lambda_2}{e^{-r_2 t}}$ and clearing for $\dot{\lambda}^1_{11}$, we can compare the dynamics of the co-state variable in the base case, eq. (6), with the co-state variable in the multi-product case that is:

$$\begin{aligned}
\dot{\lambda}^1_{11} =& r_1 \lambda^1_{11} - (q_{11} + Q_{11} x_{11}) \\
&+ \Big[\frac{(\sigma_{11})^2}{2B_{11}} + \frac{(\sigma_1)^2}{2E_1}\Big](\lambda^1_{11})^2 + \frac{(\sigma_1)^2}{2E_1}\lambda^1_{11}\lambda^1_{12}\sqrt{\frac{1 - x_{12}}{1 - x_{11}}} \\
&- \Big[\frac{(\sigma_{21})^2}{2B_{21}} + \frac{(\sigma_2)^2}{2E_2}\Big]\lambda^1_{11}\lambda^2_{11} - \frac{(\sigma_2)^2}{2E_2}\lambda^1_{11}\lambda^2_{12}\sqrt{\frac{x_{12}}{x_{11}}} \\
&- \frac{\lambda^1_{11} e_1 \sigma_1}{2E_1\sqrt{1 - x_{11}}} - \frac{\lambda^1_{11} b_{11} \sigma_{11}}{2B_{11}\sqrt{1 - x_{11}}} - \frac{\lambda^1_{11} e_2 \sigma_2}{2E_2\sqrt{x_{11}}} - \frac{\lambda^1_{11} b_{21} \sigma_{21}}{2B_{21}\sqrt{x_{11}}}
\end{aligned} \tag{41}$$

Clearly, in this case $\dot{\lambda}^1_{11}$ depend explicitly on the state variables x_{11} and x_{12}, as opposed to the aggregate case. Finally, as in the general model, the equations (28), (35), (36), (37), (38), (39) and (40) give origin to a systems of equations of six dimensions, which can be integrated numerically to give the solutions to the problem. In particular, in the next sections, we will evaluate some numerical cases and compare it with a pure 2×2 global game. We started first with the simplest global dynamics in the next subsection.

4.2. **The bang-bang control for the 2 × 2 game: The global case.** In this section the bang-bang control of the duopoly game developed above for the global case will be analyzed. For greater simplicity, we will assume that $Q_{11} = 0$, $Q_{21} = 0$, $E_1 = 0$, $E_2 = 0$, $r_1 = 0$ and $r_2 = 0$ in order to found some analytical solutions. Also, we assume that the control variables are restricted to the positive intervals

$$I_1 = \{\mu_1^- \leq \mu_1 \leq \mu_1^+\} \ , \qquad I_2 = \{\mu_2^- \leq \mu_2 \leq \mu_2^+\}$$

The linear game is defined by gamers functionals

$$J_1 = \int (q_{11}x - e_1\mu_1) \ dt \tag{42}$$

and

$$J_2 = \int = (q_{21}x - e_2\mu_2) \ dt \tag{43}$$

subject to a linear form of the state variable equation given by

$$\tfrac{dx}{dt} = \alpha_1(1-x) - \alpha_2 x$$

where

$$\alpha_1 = \sigma_1\mu_1 \ , \qquad \alpha_2 = \sigma_2\mu_2$$

The respective Hamiltonians are

$$H^1 = (q_{11}x - e_1\mu_1) + \lambda^1(\sigma_1\mu_1(1-x) - \sigma_2\mu_2 x)$$
$$H^2 = (q_{21}(1-x) - e_2\mu_2) + \lambda^2(\sigma_1\mu_1(1-x) - \sigma_2\mu_2 x)$$

or

$$H^1 = q_{11}x + [\lambda^1\sigma_1(1-x) - e_1]\mu_1 - [\lambda^1\sigma_2 x]\mu_2$$
$$H^2 = q_{21}(1-x) + [\lambda^2\sigma_1(1-x)]\mu_1 - [\lambda^2\sigma_2 x + e_2]\mu_2$$

or

$$H^1 = q_{11}x + m_1^1 \ \mu_1 + m_2^1 \ \mu_2$$
$$H^2 = q_{21}(1-x) + m_1^2 \ \mu_1 + m_2^2 \ \mu_2$$

Hence, the Hamiltonians are planes in the (μ_1, μ_2) space. The optimization of both Hamiltonian respect to the control variables, implies that there are in principle four possibles constant choices for the optimal (μ_1^*, μ_2^*) solution which are (they are the vertex of the square $I_1 \times I_2$):

$$(\mu_1^+, \mu_2^+), \quad (\mu_1^+, \mu_2^-), \quad (\mu_1^-, \mu_2^+), \quad (\mu_1^-, \mu_2^-).$$

The specific values of (μ_1^*, μ_2^*) are depending on the sign of the slopes m_1^1, m_2^1, m_1^2 and m_2^2. An important point is that the optimal control solution is a piece-wise time function. So the the game is characterized by time intervals in which the values of μ_1^* and μ_2^* are constant. We will denoted these constant optimal solutions generically by (μ_1^*, μ_2^*) (it specific values depend of the chosen time interval).

The Pontryagin equations for the Lagrange multipliers are

$$-\dot{\lambda}^1 = \frac{\partial H^1}{\partial x} \ , \qquad -\dot{\lambda}^2 = \frac{\partial H^2}{\partial x}$$

that is

$$-\dot{\lambda}^1(t) = q_{11} - (\sigma_1\mu_1^* + \sigma_2\mu_2^*) \ \lambda^1(t)$$
$$-\dot{\lambda}^2(t) = -q_{21} - (\sigma_1\mu_1^* + \sigma_2\mu_2^*) \ \lambda^2(t)$$

or

$$-\dot{\lambda}^1(t) = q_{11} - a^*\lambda^1(t)$$
$$-\dot{\lambda}^2(t) = -q_{21} - a^*\lambda^2(t)$$

with $a^* = (\sigma_1\mu_1^* + \sigma_2\mu_2^*)$. Note that $a^* > 0$ because μ_1^-, μ_1^+, μ_2^-, μ_2^+, σ_1 and σ_2 are positive quantities and also a^* is a piece-wise time function. If the game is defined in the interval $0 \leq t \leq T$, the solution of the above equations that satisfy the transversality conditions

$$\lambda^1(T) = 0 \ , \qquad \lambda^2(T) = 0$$

are

$$\lambda^1(t) = \frac{q_{11}}{a^*}(1 - e^{a^*(t-T)}) \tag{44}$$

$$\lambda^2(t) = -\frac{q_{21}}{a^*}(1 - e^{a^*(t-T)}) \tag{45}$$

so $\lambda^1(t) > 0$ and $\lambda^2(t) < 0$ in the interval $0 \leq t \leq T$.

The state variable satisfy the equation

$$\dot{x}(t) = \sigma_1\mu_1^*(1 - x(t)) - \sigma_2\mu_2^* x(t)$$

or

$$\dot{x}(t) = \sigma_1\mu_1^* - a^* x(t)$$

The form of solution for the initial condition $x(0) = x_0$ in a specific piece-wise time interval is

$$x(t) = \frac{\sigma_1\mu_1^*}{a^*} - [\frac{\sigma_1\mu_1^*}{a^*} - x_0]e^{-a^* t} \tag{46}$$

where μ_1^* is a piecewise function in time.

The stationary solution for this game are

$$-\dot{\lambda}^1(t) = q_{11} - a^*\lambda^1(t) = 0$$

$$-\dot{\lambda}^2(t) = -q_{21} - a^*\lambda^2(t) = 0$$

$$\dot{x}(t) = \sigma_1\mu_1^* - a^* x(t) = 0$$

that is

$$\lambda^{1s} = \frac{q_{11}}{a^*} \ , \qquad \lambda^{2s} = -\frac{q_{21}}{a^*} \ , \qquad x^s = \frac{\sigma_1\mu_1^*}{a^*}.$$

For a finite time game horizon T these static solutions are never reached for this system. Only if $T \to \infty$, these stationary point are reached from the solutions as one can see from (49), (50) and (51). In fact, for the infinite-horizon game the Lagrange multipliers are constant and given by

$$\lambda^1(t) = \frac{q_{11}}{a^*} \qquad t \geq 0$$

$$\lambda^2(t) = -\frac{q_{21}}{a^*} \qquad t \geq 0$$

The dynamical evolution of the system through of the different piece-wise sectors, will depend on the specific chosen values of the game parameters.

As an example, and for greater simplicity, we will assume that $Q_{11} = 0$, $Q_{21} = 0$, $E_1 = 0$, $E_2 = 0$, $r_1 = 0$ and $r_2 = 0$ and we will set all other parameters equal to one, in order to examine the simple singular solution. Also, we will assume that the control variables are restricted to the following intervals:

$$I_1 = \{0 \leq \mu_1 \leq 1\} \ , \qquad I_2 = \{0 \leq \mu_2 \leq 1\}$$

Thus, the linear game is defined by gamers functionals:

$$J_1 = \int_0^T x - \mu_1 \; dt \tag{47}$$

and

$$J_2 = \int_0^T (1-x) - \mu_2 \; dt \tag{48}$$

subject to a linear form of the state variable equation given by

$$\tfrac{dx}{dt} = (1-x)\mu_1 - x\mu_2$$

The respective Hamiltonians are

$$H^1 = (x - \mu_1) + \lambda^1\Big((1-x)\mu_1 - x\mu_2\Big)$$

$$H^2 = ((1-x) - \mu_2) + \lambda^2\Big((1-x)\mu_1 - x\mu_2\Big)$$

or

$$H^1 = x + [\lambda^1(1-x) - 1]\mu_1 - [\lambda^1 x]\mu_2$$
$$H^2 = (1-x) + [\lambda^2(1-x)]\mu_1 - [\lambda^2 x + 1]\mu_2$$

or

$$H^1 = x + m^1_1 \; \mu_1 + m^1_2 \; \mu_2$$
$$H^2 = (1-x) + m^2_1 \; \mu_1 + m^2_2 \; \mu_2$$

Hence, the Hamiltonians are planes in the (μ_1, μ_2) space. The optimization of both Hamiltonian respect to the control variables, implies that there are in principle four possibles constant choices for the optimal (μ_1^*, μ_2^*) solution which are (they are the vertex of the square $I_1 \times I_2$):

$$(1,1), \quad (0,1), \quad (1,0), \quad (0,0).$$

The specific values of (μ_1^*, μ_2^*) are depending on the sign of the slopes m^1_1, m^1_2, m^2_1 and m^2_2. An important point is that the optimal control solution is a piece-wise time function. So the the game is characterized by time intervals in which the values of μ_1^* and μ_2^* are constant. We will denoted these constant optimal solutions generically by (μ_1^*, μ_2^*) (it specific values depend of the chosen time interval).

The Pontryagin equations for the Lagrange multipliers are

$$-\dot{\lambda}^1 = \frac{\partial H^1}{\partial x} \;, \qquad -\dot{\lambda}^2 = \frac{\partial H^2}{\partial x}$$

that is

$$-\dot{\lambda}^1(t) = 1 - (\mu_1^* + \mu_2^*) \; \lambda^1(t)$$
$$-\dot{\lambda}^2(t) = -1 - (\mu_1^* + \mu_2^*) \; \lambda^2(t)$$

or

$$-\dot{\lambda}^1(t) = 1 - a^* \lambda^1(t)$$
$$-\dot{\lambda}^2(t) = -1 - a^* \lambda^2(t)$$

with $a^* = (\mu_1^* + \mu_2^*)$. Note that $a^* \geq 0$ and also a^* is a piece-wise time function. If the game is defined in the interval $0 \leq t \leq T$, the solution of the above equations that satisfy the transversality conditions

$$\lambda^1(T) = 0 \;, \qquad \lambda^2(T) = 0$$

are

$$\lambda^1(t) = \frac{1}{a^*}(1 - e^{a^*(t-T)}) \tag{49}$$

$$\lambda^2(t) = -\frac{1}{a^*}(1 - e^{a^*(t-T)}) \tag{50}$$

so $\lambda^1(t) > 0$ and $\lambda^2(t) < 0$ in the interval $0 \leq t \leq T$.

The state variable satisfy the equation

$$\dot{x}(t) = \mu_1^*(1 - x(t)) - \mu_2^* x(t)$$

or

$$\dot{x}(t) = \mu_1^* - a^* x(t)$$

The form of solution for the initial condition $x(0) = x_0$ in a specific piece-wise time interval is

$$x(t) = \frac{\mu_1^*}{a^*} - \left[\frac{\mu_1^*}{a^*} - x_0\right] e^{-a^* t} \tag{51}$$

where μ_1^* is a piecewise function in time. The intuition behin this very simplified game is clear, the market share of a firm in a particular market will be directly proportional to its relative marketing effort: $\frac{\mu_1^*}{a^*} = \frac{\mu_1^*}{\mu_1^* + \mu_2^*}$.

The stationary solution for this game are

$$-\dot{\lambda}^1(t) = 1 - a^* \lambda^1(t) = 0$$

$$-\dot{\lambda}^2(t) = -1 - a^* \lambda^2(t) = 0$$

$$\dot{x}(t) = \mu_1^* - a^* x(t) = 0$$

that is

$$\lambda^{1s} = \frac{1}{a^*}\ , \qquad \lambda^{2s} = -\frac{1}{a^*}\ , \qquad x^s = \frac{\mu_1^*}{a^*}.$$

For a finite time game horizon T these static solutions are never reached for this system. Only if $T \to \infty$, these stationary point are reached from the solutions as one can see from (49), (50) and (51). In fact, for the infinite-horizon game the Lagrange multipliers are constant and given by

$$\lambda^1(t) = \frac{1}{a^*} \qquad t \geq 0$$

$$\lambda^2(t) = -\frac{1}{a^*} \qquad t \geq 0$$

The dynamical evolution of the system through of the different piece-wise sectors, will depend on the specific chosen values of the game parameters.

4.3. **The bang-bang control for the 2 × 2 game: The local case.** We assume that the control variables are restricted to the positive intervals

$$I_{1k} = \{\mu_{1k}^- \leq \mu_{1k} \leq \mu_{1k}^+\}\ , \qquad I_{2k} = \{\mu_{2k}^- \leq \mu_{2k} \leq \mu_{1k}^+\},$$
$$I_1 = \{\mu_1^- \leq \mu_1 \leq \mu_1^+\}\ , \qquad I_2 = \{\mu_2^- \leq \mu_2 \leq \mu_2^+\}$$

In this case the linear game is defined by the functionals

$$J_1 = \int_0^T \left[\sum_{k=1}^2 (q_{1k}\ x_{1k}(t) - b_{1k}\mu_{1k}(t)) - e_1\mu_1(t)\right] dt \tag{52}$$

$$J_2 = \int_0^T \left[\sum_{k=1}^{2} (q_{2k}\ (1 - x_{1k}(t)) - b_{2k}\mu_{2k}(t)) - e_2\mu_2(t) \right] dt \tag{53}$$

subject to a linear form of the state variables equation given by

$$\dot{x}_{1k} = (\sigma_{1k}\mu_{1k} + \sigma_1\mu_1)(1 - x_{1k}) - (\sigma_{2k}\mu_{2k} + \sigma_2\mu_2)\ x_{1k} \qquad k = 1, 2 \tag{54}$$

The respective Hamiltonians are

$$\begin{aligned} H^1 = {} & \sum_{k'=1}^{2} \Big(\ q_{1k'} x_{1k'}(t) - b_{1k'}\mu_{1k'}(t) \Big) - e_1\mu_1(t) \\ & + \sum_{k'=1}^{2} \lambda^1_{k'} \Big(\alpha_{1k'}(t)(1 - x_{1k'}(t)) - \alpha_{2k'}(t) x_{1k'}(t) \Big) \end{aligned} \tag{55}$$

$$\begin{aligned} H^2 = {} & \sum_{k'=1}^{2} \Big(\ q_{2k'}(1 - x_{1k'}(t)) - b_{2k'}\mu_{2k'}(t) \Big) - e_2\mu_2(t) \\ & + \sum_{k'=1}^{2} \lambda^2_{k'} \Big(\alpha_{1k'}(t)(1 - x_{1k'}(t)) - \alpha_{2k'}(t) x_{1k'}(t) \Big) \end{aligned} \tag{56}$$

Again the Hamiltonians H^1 and H^2 are hyperplanes in the six-dimensional space of control variables $(\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}, \mu_1, \mu_2)$ of the form

$$H^1 = c^1 + \sum_{k=1}^{2} m^1_{1k}\mu_{1k} + \sum_{k=1}^{2} m^1_{2k}\mu_{2k} + m^1_1\mu_1 + m^1_2\mu_2, \tag{57}$$

$$H^2 = c^2 + \sum_{k=1}^{2} m^2_{1k}\mu_{1k} + \sum_{k=1}^{2} m^2_{2k}\mu_{2k} + m^2_1\mu_1 + m^2_2\mu_2, \tag{58}$$

with

$$c^1 = \sum_{k=1}^{2}\ q_{1k} x_{1k}(t)\ , \qquad c^2 = \sum_{k=1}^{2}\ q_{2k}(1 - x_{1k}(t))$$

$$m^1_{1k} = \sum_{k=1}^{2} \Big[-b_{1k} + \lambda^1_k \sigma_{1k}(1 - x_{1k}) \Big]\ , \qquad m^2_{1k} = \sum_{k=1}^{2} \Big[\lambda^2_k \sigma_{1k}(1 - x_{1k}) \Big]$$

$$m^1_{2k} = -\sum_{k=1}^{2} \Big[\lambda^1_k \sigma_{2k} x_{1k} \Big]\ , \qquad m^2_{2k} = -\sum_{k=1}^{2} \Big[b_{2k} + \lambda^2_k \sigma_{2k} x_{1k} \Big]$$

$$m^1_1 = -e_1 + \sigma_1 \sum_{k=1}^{2} \Big[\lambda^1_k (1 - x_{1k}) \Big] \qquad m^2_1 = \sigma_1 \sum_{k=1}^{2} \Big[\lambda^2_k (1 - x_{1k}) \Big]$$

$$m^1_2 = -\sigma_2 \sum_{k=1}^{2} \Big[\lambda^1_k x_{1k} \Big] \qquad m^2_2 = -e_1 - \sigma_2 \sum_{k=1}^{2} \Big[\lambda^2_k x_{1k} \Big]$$

In this case, the optimal solution for the control variables would be in the vertex of a six dimensional hypercube $C = I_{11} \times I_{12} \times I_{21} \times I_{22} \times I_1 \times I_2$. The specific optimal values will depend on the sign of the slopes m^1_{1k}, m^1_1, m^2_{2k} and m^2_{2k} in the Hamiltonian hyperplanes, and these slopes depends on the particular chosen values of the game's parameters. But whatever be the specific optimal control values, one can obtain the structural form of the game solution which is valid in any sector of the optimal control parameter space. Thus, the optimal control variables will be again a piecewise time functions and the solution of the game will have the same structural form in each of these piecewise time sector, differing only in the specific optimal control values. In fact, the Pontryagin equations for the Lagrangian multipliers are

$$-\dot{\lambda}^1_k = \frac{\partial H^1}{\partial x_{1k}} \ , \qquad -\dot{\lambda}^2_k = \frac{\partial H^2}{\partial x_{2k}}$$

that is

$$-\dot{\lambda}^1_k(t) = q_{1k} - \Big[\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1 + \sigma_{2k}\mu^*_{2k} + \sigma_2\mu^*_2\Big]\lambda^1_k(t)$$
$$-\dot{\lambda}^2_k(t) = -q_{2k} - \Big[\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1 + \sigma_{2k}\mu^*_{2k} + \sigma_2\mu^*_2\Big]\lambda^2_k(t)$$

or

$$-\dot{\lambda}^1_k(t) = q_{1k} - a^*_k\lambda^1_k(t)$$
$$-\dot{\lambda}^2_k(t) = -q_{2k} - a^*_k\lambda^2_k(t)$$

with $a^*_k = (\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1 + \sigma_{2k}\mu^*_{2k} + \sigma_2\mu^*_2)$. Note that again $a^*_k > 0$ because μ^-_{1k}, μ^+_{1k}, μ^-_{2k}, μ^+_{2k}, μ^-_1, μ^+_1, μ^-_2, μ^+_2, σ_{1k}, σ_{2k}, σ_1 and σ_2 are positive quantities and also a^*_k is a piece-wise time function. If the game is defined in the interval $0 \leq t \leq T$, the solution of the above equations that satisfy the transversality conditions

$$\lambda^1_k(T) = 0 \ , \qquad \lambda^2_k(T) = 0$$

are

$$\lambda^1_k(t) = \frac{q_{1k}}{a^*_k}(1 - e^{a^*_k(t-T)}) \tag{59}$$

$$\lambda^2_k(t) = -\frac{q_{2k}}{a^*_k}(1 - e^{a^*_k(t-T)}) \tag{60}$$

so $\lambda^1_k(t) > 0$ and $\lambda^2_k(t) < 0$ in the interval $0 \leq t \leq T$.

The state variable satisfy the equation (54) or

$$\dot{x}_k(t) = (\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1) - a^*_k x_k(t)$$

The form of solution for the initial condition $x_k(0) = x^0_k$ in a specific piece-wise time interval is

$$x_k(t) = \frac{(\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1)}{a^*_k} - \left[\frac{(\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1)}{a^*_k} - x^0_k\right]e^{-a^*_k t} \tag{61}$$

where μ^*_{1k}, μ^*_1 and a_k are piecewise functions of time.

The stationary solution for this local game are

$$\lambda^{1\ s}_k = \frac{q_{1k}}{a^*_k} \ , \qquad \lambda^{2\ s}_k = -\frac{q_{2k}}{a^*_k} \ , \qquad x^s_k = \frac{\sigma_{1k}\mu^*_{1k} + \sigma_1\mu^*_1}{a^*_k}.$$

For a finite time game horizon T these static solutions are never reached for this system. Only if $T \to \infty$, again these stationary point are reached from the solutions as one can see from (59), (60) and (61). In fact, for the infinite-horizon game the Lagrange multipliers are constant and given by

$$\lambda^1_k(t) = \frac{q_{1k}}{a^*_k} \qquad t \geq 0$$

$$\lambda^2_k(t) = -\frac{q_{2k}}{a^*_k} \qquad t \geq 0$$

The dynamical evolution of the system through of the different piece-wise sectors, will depend on the specific chosen values of the game parameters.

4.4. **The pure global case for the 2×2 game.** In this section an analysis of the pure global case will be developed and then it will be compared with the local case. Consider a two firm pure global game ($i=1,2$ and $k=1$) whose state variables are $x_{11}(t)$ and $x_{21}(t)$ for each player respectively. These variables are constrained to $x_{11}(t)+x_{21}(t)=1$ for all $t\,\epsilon\,[0,\ T]$. We choose $x(t)=x_{11}(t)$ as the independent state variable, so $x_{21}=1-x(t)$. The dynamical behavior of the independent variable is given by an equation analogous to equation (28) but taken $\sigma_{ik}=0,\ b_{ik}=0$ and $B_{ik}=0$ for $i=1,\ 2,\ k=1,\ 2$, that is

$$\dot{x}(t)=\alpha_1(t)\sqrt{1-x(t)}-\alpha_2(t)\sqrt{x(t)} \tag{62}$$

where

$$\alpha_1(t)=\sigma_1\ \mu_1(t) \tag{63}$$

$$\alpha_2(t)=\sigma_2\ \mu_2(t) \tag{64}$$

and corresponds to the global part of the α_i. The players functionals are in this global case

$$J_1=\int_0^T e^{-r_1 t}\left[q_{11}x(t)+\frac{1}{2}Q_{11}x(t)^2-e_1\mu_1(t)-\frac{1}{2}\ E_1\ (\mu_1(t))^2\right]\,dt \tag{65}$$

and

$$J_2=\int_0^T e^{-r_2 t}\left[q_{21}(1-x(t))+\frac{1}{2}Q_{21}(1-x(t))^2-e_2\mu_2(t)-\frac{1}{2}\ E_2\ (\mu_2(t))^2\right]\,dt \tag{66}$$

subject to the equation (62). The corresponding Hamiltonians are (here $\lambda^1=\lambda^1_{11}$ and $\lambda^2=\lambda^2_{11}$)

$$\begin{aligned}H^1=e^{-r_1 t}&\left[q_{11}x(t)+\frac{1}{2}Q_{11}x(t)^2-e_1\mu_1(t)-\frac{1}{2}\ E_1\ (\mu_1(t))^2\right]\\&+\lambda^1\left(\alpha_1(t)\sqrt{1-x(t)}-\alpha_2(t)\sqrt{x(t)}\right)\end{aligned} \tag{67}$$

and

$$\begin{aligned}H^2=e^{-r_2 t}&\left[q_{21}(1-x(t))+\frac{1}{2}Q_{21}(1-x(t))^2-e_2\mu_2(t)-\frac{1}{2}\ E_2\ (\mu_2(t))^2\right]\\&\lambda^2\left(\alpha_1(t)\sqrt{1-x(t)}-\alpha_2(t)\sqrt{x(t)}\right)\end{aligned} \tag{68}$$

The Pontryagin equations give for the optimal controls

$$\mu_1(t)=\frac{\lambda^1\ \sigma_1\ \sqrt{1-x(t)}}{e^{-r_1 t}\ E_1}-\frac{e_1}{E_1} \tag{69}$$

$$\mu_2(t)=-\frac{\lambda^2\ \sigma_2\ \sqrt{x(t)}}{e^{-r_2 t}\ E_2}-\frac{e_2}{E_2} \tag{70}$$

and the dynamical equations for the Lagrangian multipliers are

$$\dot{\lambda}^1 = -e^{-r_1 t}(q_{11} + Q_{11}x(t)) + \lambda^1\left(\frac{\alpha_1}{2\sqrt{1-x}} + \frac{\alpha_2}{2\sqrt{x}}\right) \tag{71}$$

and

$$\dot{\lambda}^2 = e^{-r_2 t}(q_{21} + Q_{21}(1-x(t))) + \lambda^2\left(\frac{\alpha_1}{2\sqrt{1-x}} + \frac{\alpha_2}{2\sqrt{x}}\right) \tag{72}$$

or by replacing the explicit form of the controls (69), (70), one obtains finally

$$\begin{aligned}\dot{\lambda}^1 =& -e^{-r_1 t}(q_{11} + Q_{11}x(t)) + \frac{(\lambda^1)^2(\sigma_1)^2}{2E_1 e^{-r_1 t}} \\ & - \frac{\lambda^1\lambda^2(\sigma_2)^2}{2E_2 e^{-r_2 t}} - \frac{\lambda^1\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\lambda^1\sigma_2 e_2}{2E_2\sqrt{x}}\end{aligned} \tag{73}$$

and

$$\begin{aligned}\dot{\lambda}^2 =& e^{-r_2 t}(q_{21} + Q_{21}(1-x(t))) - \frac{(\lambda^2)^2(\sigma_2)^2}{2E_2 e^{-r_2 t}} \\ & + \frac{\lambda^1\lambda^2(\sigma_1)^2}{2E_1 e^{-r_1 t}} - \frac{\lambda^2\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\lambda^2\sigma_2 e_2}{2E_2\sqrt{x}}\end{aligned} \tag{74}$$

Equations (73) and (74) gives a "Ricatti like" system of coupled equations that depend explicitly on $x(t)$. Only if the linear terms of the global controls μ_1 and μ_2 and the quadratic terms in the state variable $x(t)$ does not appear in the Hamiltonian's players (that is, $e_1 = 0$, $e_2 = 0$, $Q_{11} = 0$ and $Q_{21} = 0$), the pure global case dynamic of the Lagrangian multipliers are independent of the state variable $x(t)$ and satisfy Ricatti equations of motion.

By defining $\lambda^i = \bar{\lambda}^i e^{-r_i t}, \quad i = 1, 2$ equations (62), (73), (74) can be written in terms of $\bar{\lambda}$ as

$$\dot{x}(t) = \frac{(\sigma_1)^2}{E_1}\bar{\lambda}^1(1-x) + \frac{(\sigma_2)^2}{E_2}\bar{\lambda}^2 x - \frac{\sigma_1 e_1}{E_1}\sqrt{1-x} + \frac{\sigma_2 e_2}{E_2}\sqrt{x} \tag{75}$$

$$\dot{\bar{\lambda}}^1 = r_1\bar{\lambda}^1 - (q_{11} + Q_{11}x) + \frac{(\bar{\lambda}^1)^2(\sigma_1)^2}{2E_1} - \frac{\bar{\lambda}^1\bar{\lambda}^2(\sigma_2)^2}{2E_2} - \frac{\bar{\lambda}^1\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\bar{\lambda}^1\sigma_2 e_2}{2E_2\sqrt{x}} \tag{76}$$

$$\dot{\bar{\lambda}}^2 = r_2\bar{\lambda}^2 + (q_{21} + Q_{21}(1-x)) - \frac{(\bar{\lambda}^2)^2(\sigma_2)^2}{2E_2} + \frac{\bar{\lambda}^1\bar{\lambda}^2(\sigma_1)^2}{2E_1} - \frac{\bar{\lambda}^2\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\bar{\lambda}^2\sigma_2 e_2}{2E_2\sqrt{x}} \tag{77}$$

Thus, the global dynamic of the 2×2 game is defined by equations (75), (76) and (77).

One important limit behavior of the this systems corresponds to the static limit, in which

$$\dot{x} = 0, \qquad \dot{\bar{\lambda}}^i = 0 \quad i = 1, 2. \tag{78}$$

Then, the stationary values of the Lagrangian multipliers $\bar{\lambda}^i$ and state variable x is given by the solutions of the algebraic equations:

$$0 = \frac{(\sigma_1)^2}{E_1}\bar{\lambda}^1(1-x) + \frac{(\sigma_2)^2}{E_2}\bar{\lambda}^2 x - \frac{\sigma_1 e_1}{E_1}\sqrt{1-x} + \frac{\sigma_2 e_2}{E_2}\sqrt{x} \tag{79}$$

$$0 = r_1\bar{\lambda}^1 - (q_{11} + Q_{11}x) + \frac{(\bar{\lambda}^1)^2(\sigma_1)^2}{2E_1} - \frac{\bar{\lambda}^1\bar{\lambda}^2(\sigma_2)^2}{2E_2} - \frac{\bar{\lambda}^1\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\bar{\lambda}^1\sigma_2 e_2}{2E_2\sqrt{x}} \tag{80}$$

$$0 = r_2\bar{\lambda}^2 + (q_{21} + Q_{21}(1-x)) - \frac{(\bar{\lambda}^2)^2(\sigma_2)^2}{2E_2} + \frac{\bar{\lambda}^1\bar{\lambda}^2(\sigma_1)^2}{2E_1} - \frac{\bar{\lambda}^2\sigma_1 e_1}{2E_1\sqrt{1-x}} - \frac{\bar{\lambda}^2\sigma_2 e_2}{2E_2\sqrt{x}} \tag{81}$$

These stationary values of x and $\bar{\lambda}^i$ are the asymptotic ones, that is, correspond to the values $t \to T$ for $x(t)$ and $t \to 0$ for $\bar{\lambda}(t)$. Because $\lambda^i(t) = \bar{\lambda}^i(t)e^{-r_i t}$, $\quad i = 1, 2$, the asymptotic behavior for λ^i is

$$\lim_{t \to 0} \lambda^i(t) = \lim_{t \to 0} \bar{\lambda}^i(t)e^{-r_i t} = \bar{\lambda}^i_0 \tag{82}$$

where $\bar{\lambda}^i_0$ are the stationary solutions for $\bar{\lambda}^i$. Thus, the values of the Lagrangian multipliers λ^i goes asymptotically to the values of the stationary solutions of $\bar{\lambda}^i$.

The following figures show the numerical solution of the equation (62), (73) and (74) for the values of the global game parameters listed in table 2. Figure (3) shows the temporal dependence of state variable $x(t)$. Figure (2) shows the control variables μ_1 and μ_2 versus time. Figure (3) shows the Lagrangian multipliers λ_1 and λ_2 in function of time. At last, figure (4) shows the phase diagram for the global game (λ^i versus x). The initial condition for $x(t)$ is $x(0) = 0.2$ and the time horizon is $T = 8$.

	Global parameters
q_{11}	0.8
q_{21}	0.3
Q_{11}	0.0
Q_{21}	0.0
e_1	0.0
e_2	0.0
E_1	0.5
E_2	0.3
σ_1	0.95
σ_2	1.6
r_1	0.01
r_2	0.05

TABLE 2. Data for the pure global game.

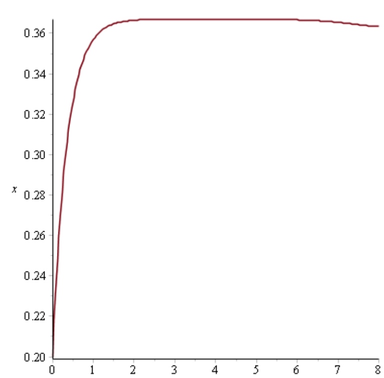

FIGURE 1. $x(t)$ for the global case in table 2

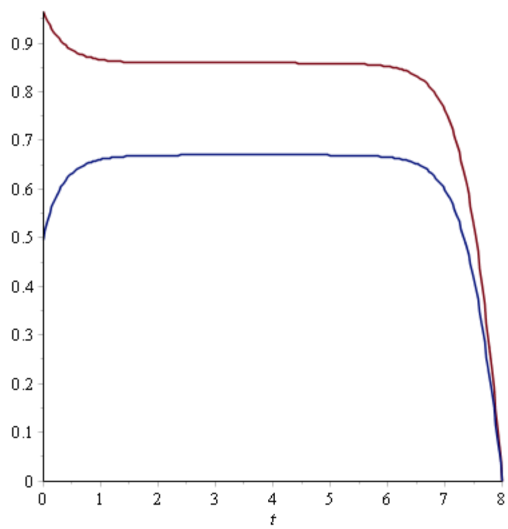

FIGURE 2. $\mu_i(t)$ for the global case in table 2

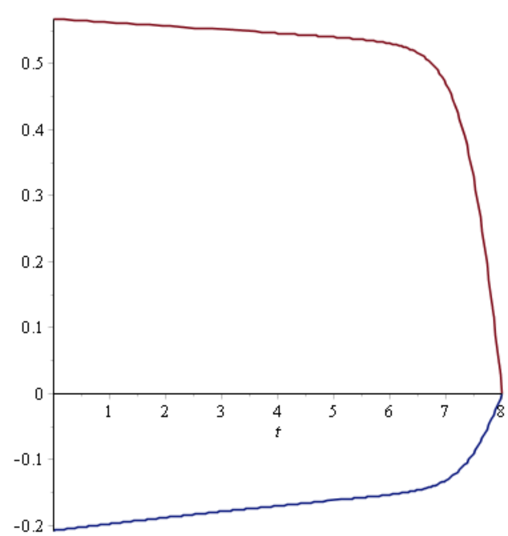

FIGURE 3. $\lambda^i(t)$ for the global case in table 2

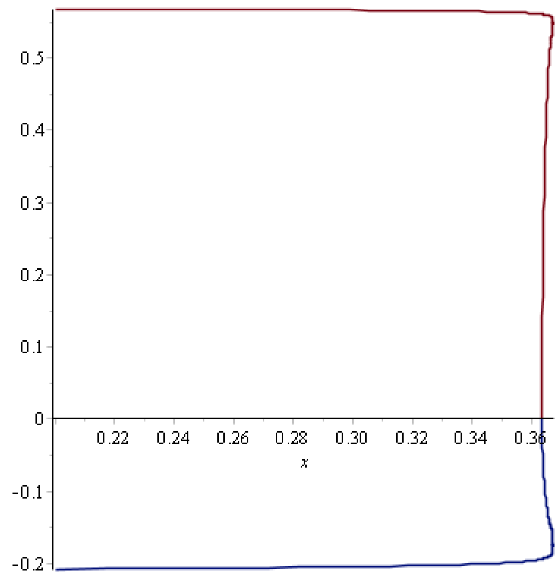

FIGURE 4. Phase space diagram (λ^i versus x) for the global case in table 2

The stationary solutions of the system (79), (80) and (81) are:

$$\begin{aligned} &1) \quad x = 0.3540, \quad \lambda_1 = -0.562, \quad \lambda_2 = 0.217 \\ &2) \quad x = 0.367, \quad \lambda_1 = 0.568, \quad \lambda_2 = -0.207 \\ &3) \quad x = 2050.099, \quad \lambda_1 = -55.474, \quad \lambda_2 = -11.728 \end{aligned} \tag{83}$$

Thus, this global game posses three different possible asymptotic stationary dynamics. For our initial conditions, the system evolves finally to the second stationary solution, as one can see clearly from the figures (1), (3), and (4).

4.5. **Results for some specific cases of the local dynamics.** In general, we have to find the following decision variables for each firm : i) the local advertising effort at each market, $\mu_{ik}(t),\ i = 1,\ 2,\ k = 1,\ 2$, and ii) the global advertising effort, $\mu_i(t),\ i = 1,\ 2$. For this particular case, there are three decision variables for each firm, one global and tow market specific, see the variables with asterisk in figure 1. On the other hand, each firm have to determine four specific parameters for each market, and other two global parameters, see figure 1. Finally, besides the decision variables, the evolution of market share $x_{ik}(t),\ i = 1,\ 2,\ k = 1,\ 2$ will be the main variable of managerial interest. For each particular Case we will also analyze the phase phase space of the system $(\lambda,\ x)$.

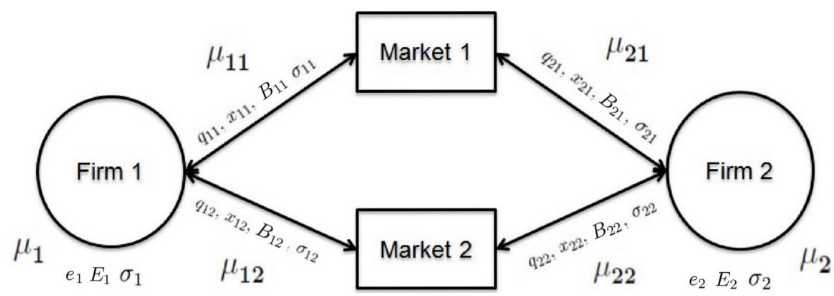

FIGURE 5. Game description

Specifically, we will develop the numerical solution and analysis of the results of four games, discussing the importance of global advertising strategies, asymmetries of the players, and the geographical presence of firms. The parameters considered are shown in table 3: (we set $Q_{ik} = 0$, $b_{ik} = 0$ and $e_i = 0$ for $i = 1, 2$ and $k = 1, 2$ all four cases below)

	Case 1		Case 2		Case 3		Case 4	
	Firm 1	Firm 2	Firm 1	Firm 2	Firm 1	Firm 2	Firm 1	Firm 2
q_{i1}	3	3	3	3	3	3	3	3
q_{i2}	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
B_{i1}	1	1	1	1	1	1	1	1
B_{i2}	0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7
E_i	1	1	1	2	1	1	1	1
σ_{i1}	0.1	0.1	0.1	0.1	0.7	0.1	0.7	0.8
σ_{i2}	0.1	0.1	0.1	0.1	0.7	0.1	0.7	0
σ_i	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2

TABLE 3. Data of numerical examples

The initial conditions for all the games are $x_{11}(0) = 0.5$, $x_{12}(0) = 0.6$, $x_{13}(0) = 0.4$. We assume that global advertising is more expensive than local efforts, however they are most effective. When firms are asymmetric, we assume that one firm has double the local advertising effectiveness. For the case with different market presence, we assume that a firm has its business in just one location, not in all markets. Finally, for all cases the discount rate will be one percent for both competitors ($r_i = 1\%$).

- **Case 1. Symmetric firms and symmetric advertising strategies: the base scenario**

From figure 2, we can see the optimal global and local advertising strategies for firm 1 and 2. Given the symmetry of this base scenario, the optimal local and global investments in advertising for firm 1 and 2 are the same. For the same reason, the market share remains the same for every firm in every market. It is important to note that since the optimal global advertising variable is non-zero for both firms, we could infer that considering the parameters of this particular example, global advertising improve profits for firms.

Figures 6 (f), 7 (f), 8 (f) and 9 (f) show the λ multipliers $\lambda_{11}, \lambda_{21}$ versus x_{11} and $\lambda_{12}, \lambda_{22}$ versus x_{12} on the same graph. Due that the standard economic interpretation of the Lagrangian multiplier is the firm's wealth increase due to one more unit of capital at time t, the phase space graphs 6 (f), 7 (f), 8 (f) and 9 (f) gives the firm's wealth increase valuation for each market in terms of the participation of firm 1 on the same market, for all cases given in table 3. From panel (f) of figure 6, we can see the phase diagram, that in this case is again symmetric. It presents different patterns for market 1 and 2, recognizing the fact that market 1 is much more profitable for unit of market share for both firms.

- **Case 2. Symmetric firms with different Global Advertising Possibilities ($E_2 = 2$).**

In this scenario, all the parameters remain the same that in case 1, so firms are still symmetric, however, the cost of global advertising is doubled for firm 2, see table 3. Given the fact that global advertising got more expensive in relative terms for firm 2, the investment for this item is reduced by firm 2, and slightly increased by firm 1. Indeed, figure 3 shows that global advertising decreased sharply in this scenario compare to that invested in case 1. On the other hand, firm 2, in order to compensate the decrease in global advertising, increased local advertising in the two markets . The results in terms of market share are shown in figure 7 panel (e) for firm 1, and hence implicitly for firm 2, since market share represents a zero-sum game. The clear results is that firm 1 increase its market share in both markets. In summary, the increase in the global advertising cost decreases market share for firm 2. In panel (f) of figure 7, we can see the phase diagram. In this case, the effects are again symmetric, but much more elastic than in the previous case, indicating the relative change in the overall cost of global advertising in favor of firm 1.

- **Case 3. Asymmetric firms ($\sigma_{1k} > \sigma_{2k}$).**

In Case 3, all the parameters remain the same that in the base scenario, nevertheless, the local advertising effectiveness for firm 1 is increased by 10%, see table 2. The main result of this example is that as a consequence of the small increase in the effectiveness of the local advertising of firm 1, this firms increase its expenditure in this item in both markets. Firm 2 does not change its global and local strategy and thus losses a small percentage of market share in market 2. In panel (f) of figure 7, we can see the phase diagram. In this case, the effects are again symmetric, but much more elastic than in the two previous cases, indicating the change in the relatively greater effectiveness in advertising of firm 1.

- **Case 4. Firms with different market presence ($\sigma_{22} = 0$).**

Finally, in this example we restrict firm 2 to a only one market, market 1. What happens is that firm one decreases its advertising efforts in market 2, and maintains its advertising rate in market 1. On the other hand, firm 2 increases its global advertising effort, since this is more cost-effective, increasing this way its market share in market 1, the only market is participating. For firm 1, this is the best scenario, since its total profit increased due to the notorious increase in market share in market 2. It is important to note, that without a model as this, developed in this paper, the traditional local optimization in a specific market could not have predicted the increase in market share in market 2 by firm 1.

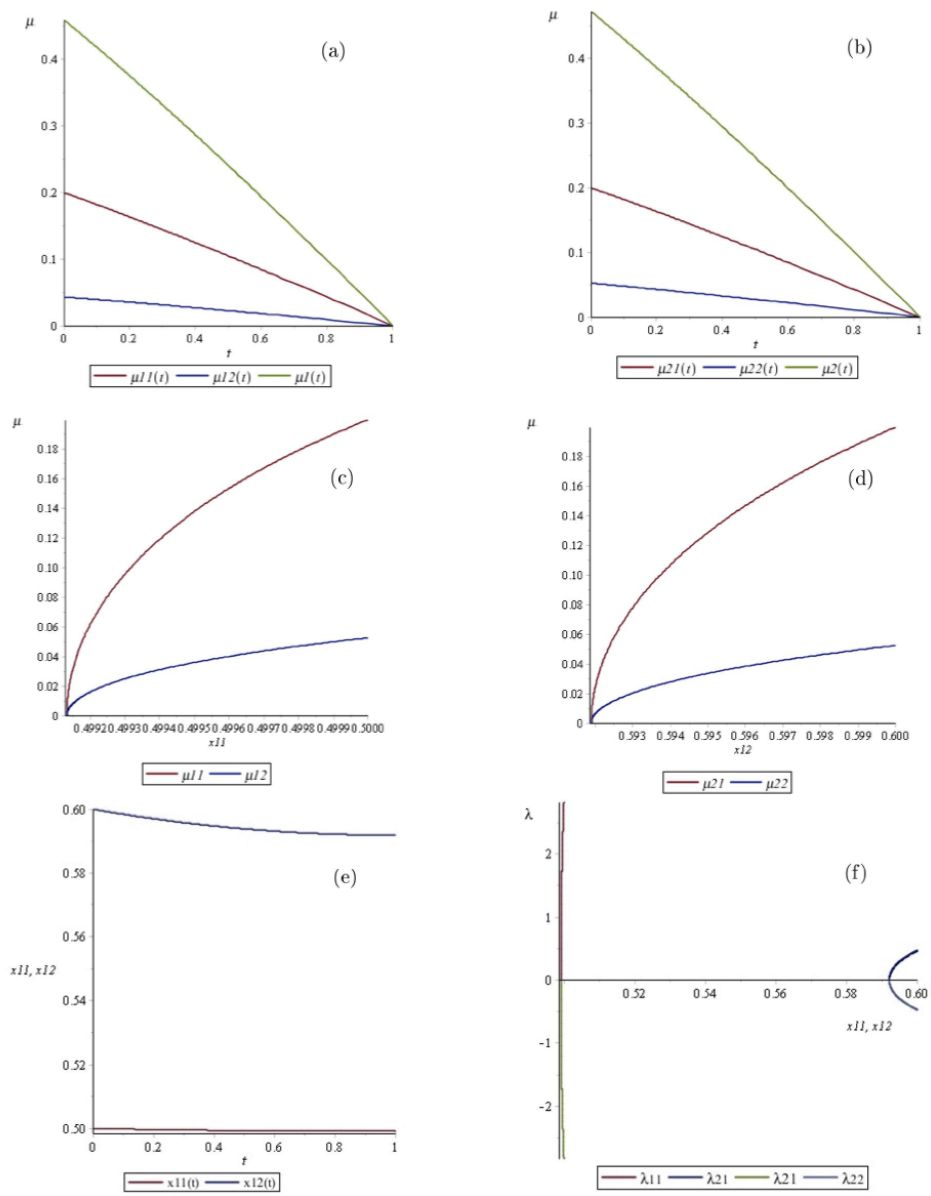

Figure 6. Case 1 in table 3

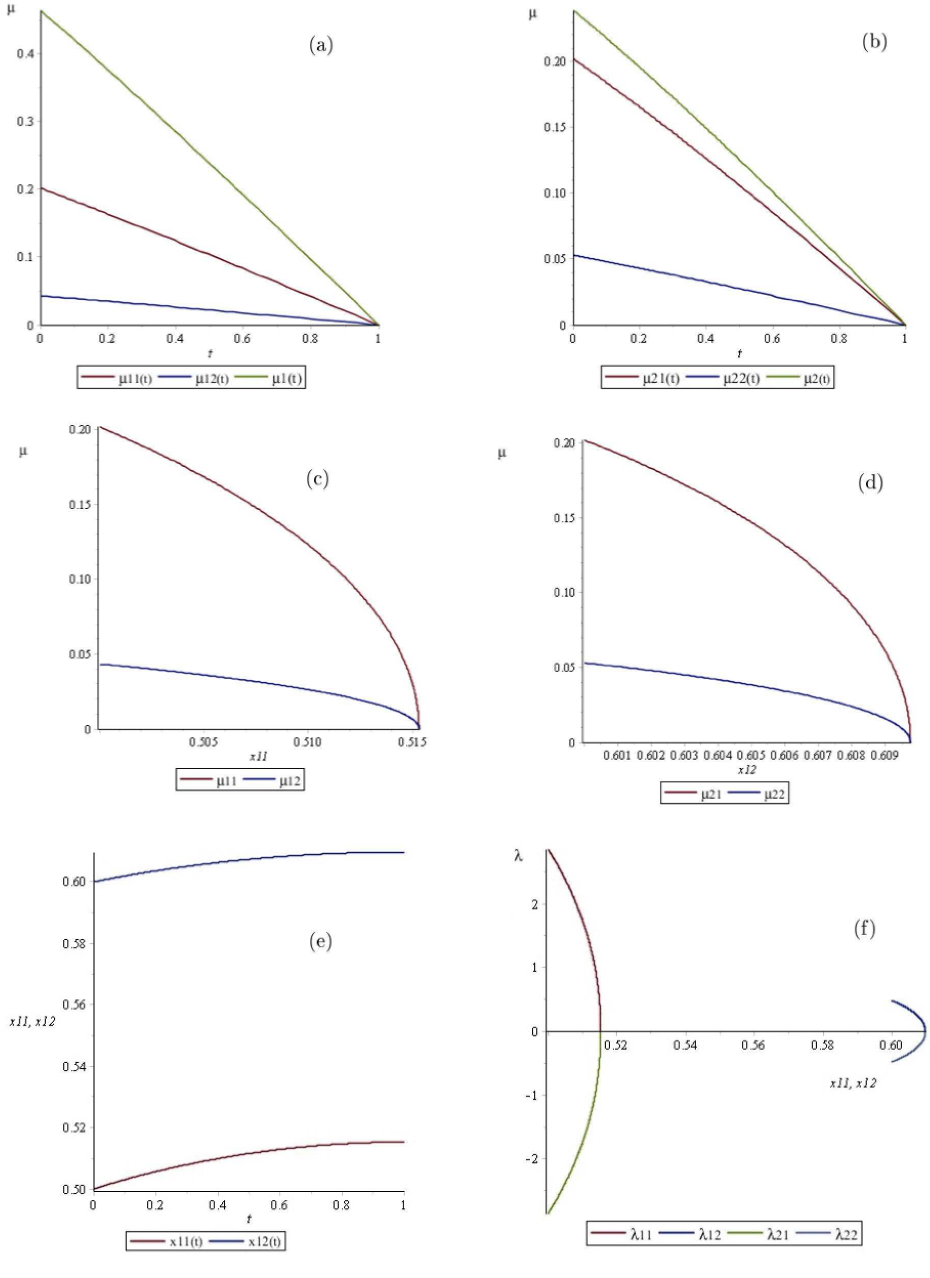

FIGURE 7. Case 2 in table 3

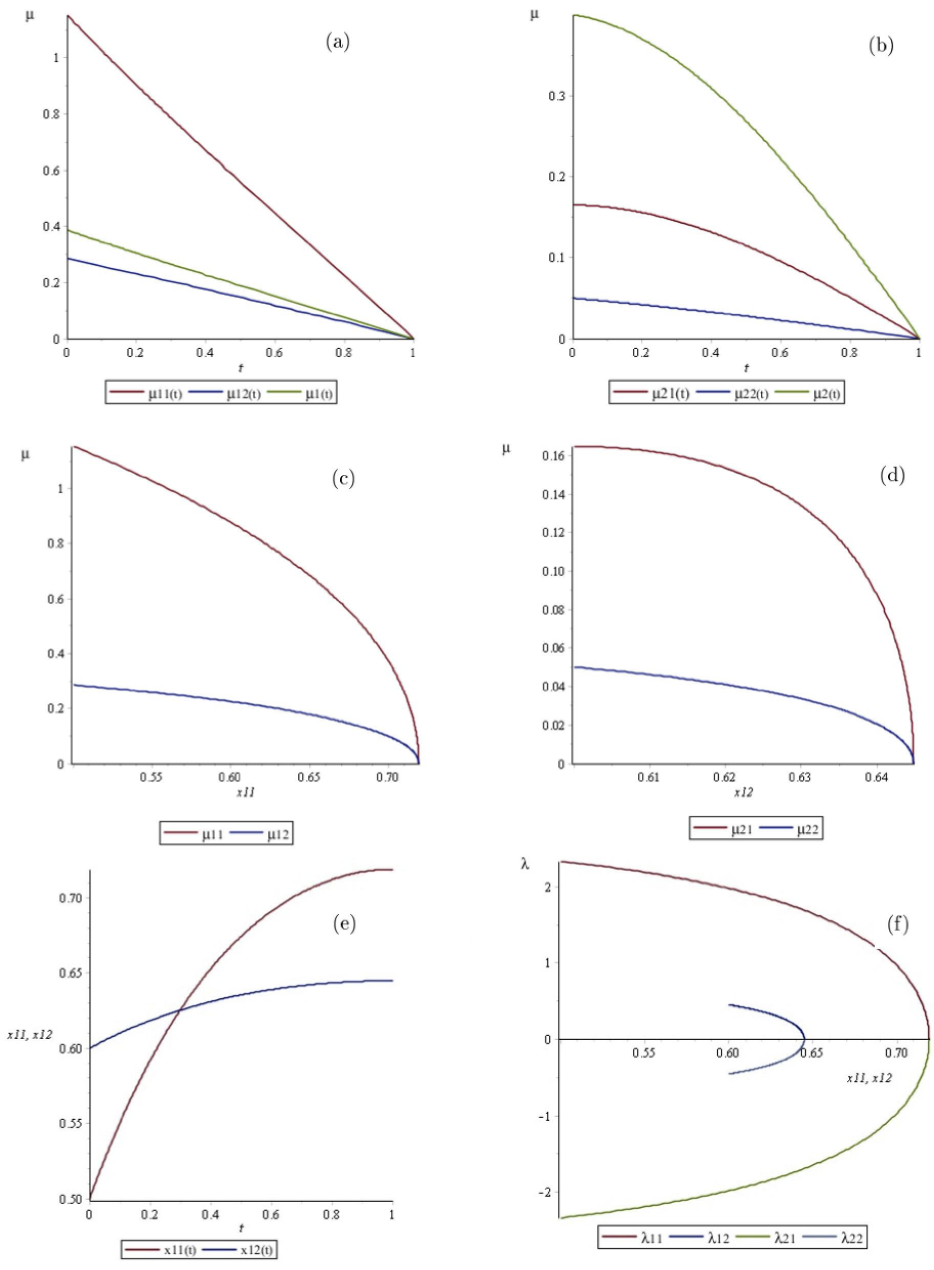

FIGURE 8. Case 3 in table 3

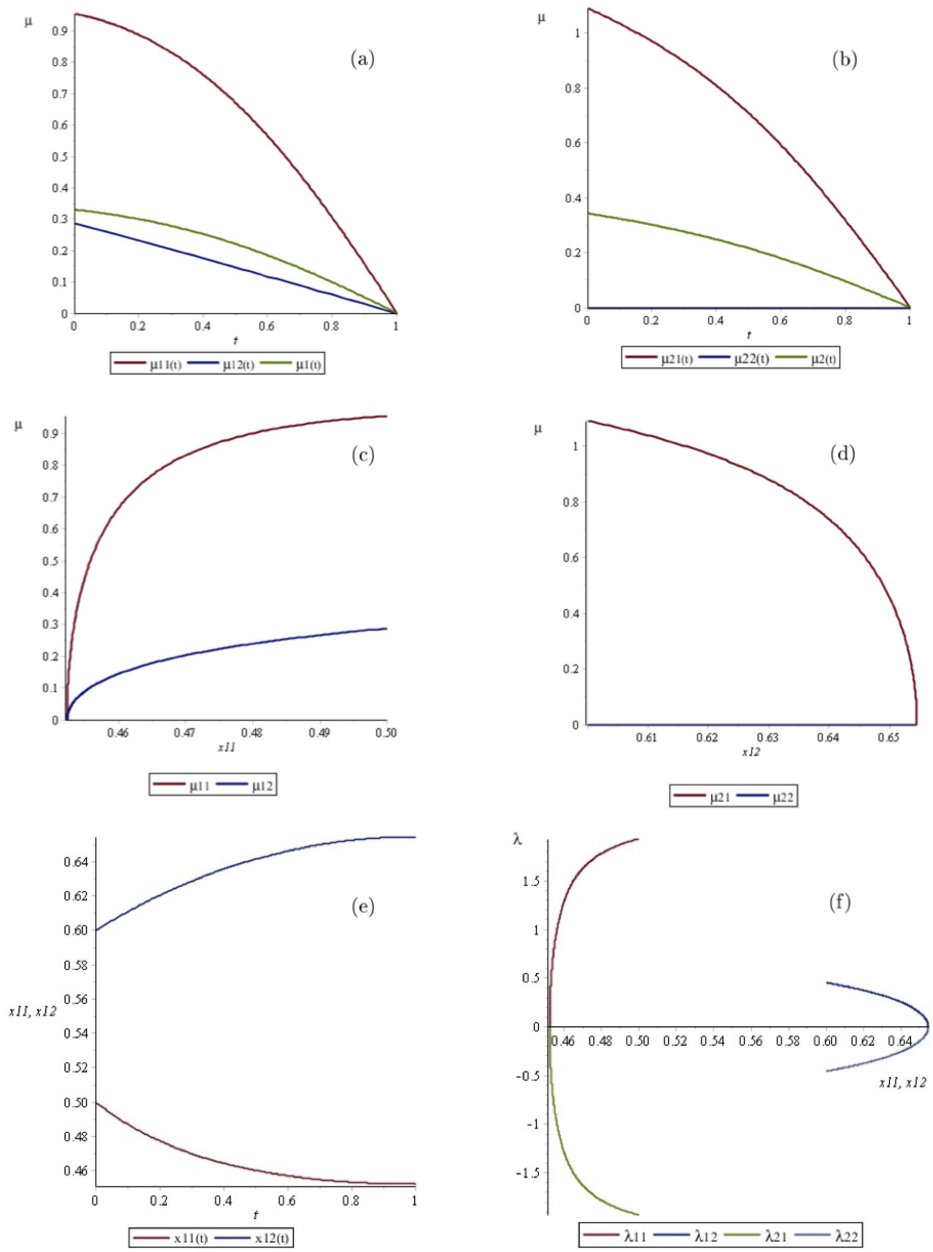

FIGURE 9. Case 4 in table 3

4.6. **Comparison between local and global dynamics for 2×2 game.** In this subsection we compare the local and global dynamics. Figure 10a shows the numerical solution of equation (62) of the pure-global game for the parameters values given in case 0 of table 4. Figure 10b shows the phase diagram for the global game (λ versus x). The initial conditions for $x(t)$ is $x(0) = 0.6$.

Now, one can use the complete local and global game to try to obtain the pure global solution given in figure 10a and explore the local effects over it. Figure 11a shows the values of $x_{11}(t)$, $x_{12}(t)$ and $x_p(t)$ (defined as the the mean value of x_{11} and x_{12}, that is, $x_p(t) = 1/2(x_{11}(t) + x_{12}(t))$) for the same "quasi-global game" given in the case 1 of table 4. Note that all they coincides for the quasi-global case and they have the same dynamical structure of the pure-global case. Figure 11b shows the phase space diagrams λ_{11}, λ_{21} versus x_{11} and λ_{12}, λ_{22} versus x_{12} in the same graph. Again, they give similar dynamics: λ_{11} is equivalent to λ_{21}, and λ_{12} is equivalent to λ_{22}. The initial conditions are $x_{11}(0) = 0.6$ and $x_{12}(0) = 0.6$

In order to explore the local effects over the global game, the figure 12 shows the solution of the complete game for the parameter values listed in the case 2 of the table 4. Figure 12a shows different evolutions for the state variables $x_{11}(t)$ and $x_{12}(t)$. The green curve is $x_p(t)$. The phase space is showed in figure 12b.

Note that the pure global game cannot "see" the local structure given by figure 12, but still the λ global multiplier has information about this structure. For example, the λ_1 global multiplier in figure 10b oscillates from the right side (from the λ_{12} side of figure 12b) to the left side (to the λ_{11} side of figure 12b). The same is truth for λ_2. The initial conditions are again $x_{11}(0) = 0.6$ and $x_{12}(0) = 0.6$

	Case 0: pure-global		Case 1: quasi-global		Case 2: local effects	
	Firm 1	Firm 2	Firm 1	Firm 2	Firm 1	Firm2
q_{i1}	0.8	0.3	0.8	0.3	0.8	0.3
q_{i2}	0.8	0.3	0.8	0.3	0.8	0.3
Q_{i1}	0	0	0	0	0	0
Q_{i2}	0	0	0	0	0	0
b_{i1}	0	0	0	0	0	0
b_{i2}	0	0	0	0	0	0
B_{i1}	0	0	0.001	0.001	5	0.1
B_{i2}	0	0	0.001	0.001	1	2
e_i	0	0	0	0	0	0
E_i	1	2	1	2	1	2
σ_{i1}	0	0	0	0	0.1	0.3
σ_{i2}	0	0	0	0	0.6	0.1
σ_i	1	1.9	1	1.9	1	1.9
r_i	0.01	0.05	0.01	0.05	0.01	0.05

TABLE 4. Data for pure-global (case 0), quasi-global (case 1) and local effects (case 2).

5. **Conclusions and further work.** Most studies found in the literature of how to model advertising strategies using differential games focus on the dynamics and market structure of the problem, but neglect their spatial dimension. Since competition in our days typically takes place on geographically separated markets, the spatial modeling of their advertising strategies becomes a crucial issue, especially in competitive multi-store retail environments.

Indeed, distinct geographical locations may involve very different realities, with highly heterogeneous parameters. Within this environment, the outcome of a general model considering average parameters may reflect sub-optimal results. On the other hand, optimal local strategies will always remain sub-optimal strategies, unless there is a perfect homogeneity between the stores' respective characteristics and environments, in terms of the effectiveness of their strategy, costs and levels of competitiveness.

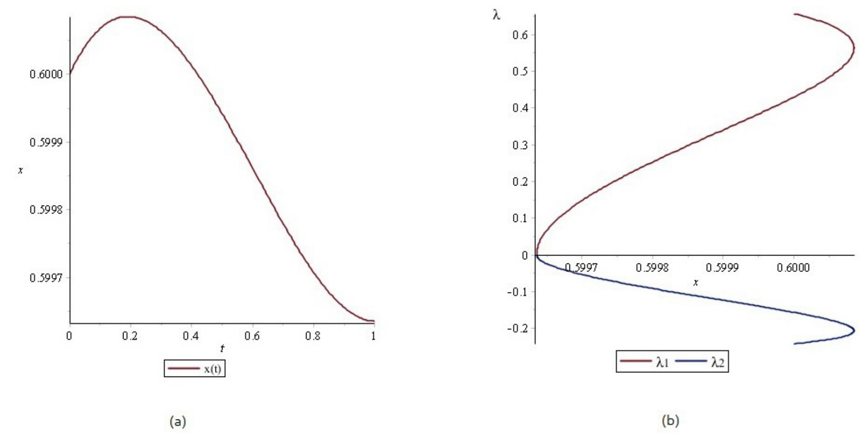

FIGURE 10. The pure-global case, case 0

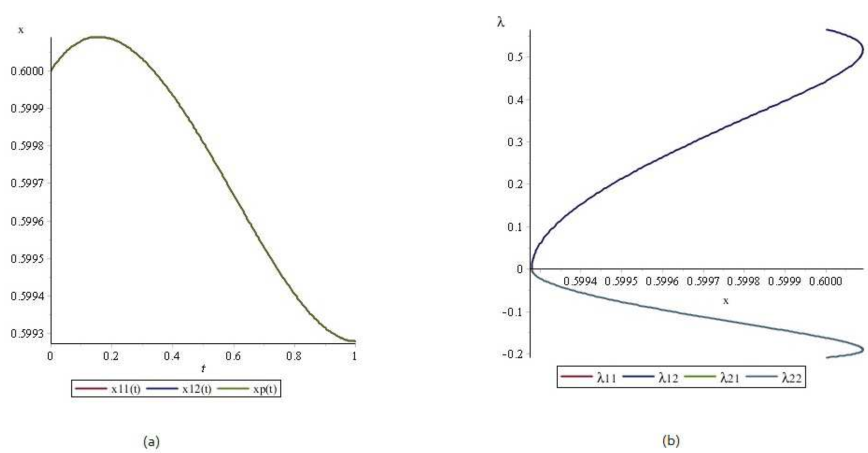

FIGURE 11. The quasi-global case, case 1

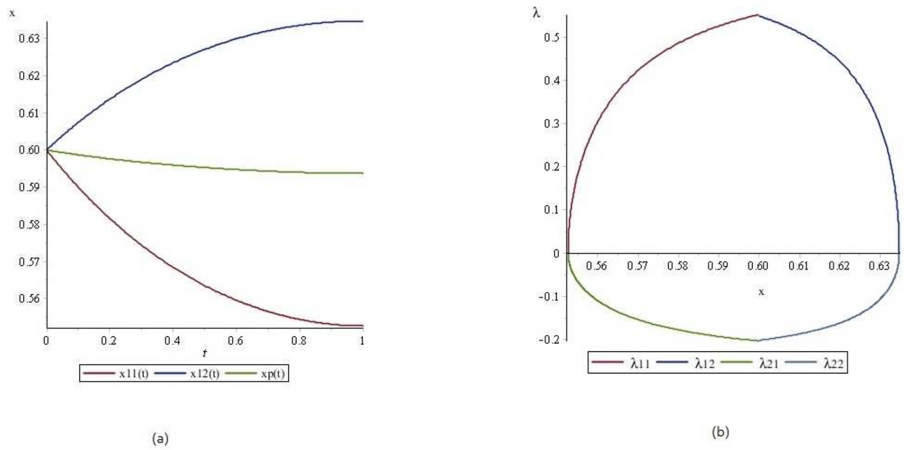

FIGURE 12. The local case, case 2

Thus, companies must usually to decide not only as to when and how much to invest in advertising, but also, where to allocate their advertising funds. Besides, retail chains have to choose whether to undertake global (i.e. worldwide, national or regional) or local (i.e. a city zone or region, or even an entire country) advertising efforts or an optimal combination of both.

In this paper, we propose a model that, while keeping the dynamic and oligopolistic characteristics of the advertising game, it introduces the spatial dimension of the problem, considering the global and local efforts of advertising. Specifically, we developed a model where companies can interact in more than one market, thereby enabling their launching of global advertising strategies influencing all markets. The model allows for the calibration of different parameters: for advertising effectiveness, markets with different levels of revenue and competition and different advertising costs for every geographical location.

The numerical results highlight the importance of spatial modeling in order to capture general equilibrium effects due to global advertising efforts and changes in specific markets that affect the others markets. These effects have significant impacts on the global profit of companies and their advertising strategies.

Firstly, we saw that global advertising could be a great advantage for multi-store competition. Indeed, a more expensive global advertising could mean a significant loss in market share, in all markets. Secondly, it is clear that changes in one specific market could affect each company's optimal equilibrium, thereby implying that aggregation using average parameters and local optimization with specific local market parameters, could result in sub-optimal policies.

Finally, it is expected that more examples that are practical may be developed based on this model, in order to prove its validity and relative importance (with respect to more aggregated previous models). Since the oligopolistic spatial markets are the rule rather than the exception, it is expected that many industries could be spatially calibrated, for instance: food chains, supermarkets chains, multinationals franchising, retailers of gasoline, banks and telecommunication, and so on.

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REFERENCES

[1] A. Beresteanu, P. Ellickson and S. Misra, The dynamics of retail oligopoly, Working Papers.
[2] M. Breton, R. Jarrar and G. Zaccour, A note on feedback sequential equilibria in a lanchester model with empirical application, *Management Science*, **52** (2006), 804–811.
[3] A. Buratto, L. Grosset and B. Viscolani, Advertising channel selection in a segmented market, *Automatica*, **42** (2006), 1343–1347.
[4] R. Cellini and L. Lambertini, Advertising in a differential oligopoly game, *Journal of Optimization Theory and Applications*, **116** (2003), 61–81.
[5] P. K. Chintagunta and D. Jain, Empirical analysis of a dynamic duopoly model of competition, *Journal of Economics and Management Strategy*, **4** (1995), 109–131.
[6] P. K. Chintagunta and N. Vilcassim, An empirical investigation of advertising strategies in a dynamic duopoly, *Management Science*, **38** (1992), 1230–1244.
[7] P. Davis, Spatial competition in retail markets: Movie theaters, *The RAND Journal of Economics*, **37** (2006), 964–982.
[8] E. J. Dockne and S. Jorgensen, New product advertising in dynamic oligopolies, *Zeitschrift fur Operations Research*, **36** (1992), 459–473.
[9] E. J. Dockner, S. Jorgensen, N. Van Long and G. Sorger, *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge, U.K, 2000.
[10] P. B. Ellickson and S. Misra, Supermarket pricing strategies, *Marketing Science*, **27** (2008), 811–828.
[11] G. M. Erickson, Empirical analysis of closed-loop duopoly advertising strategies, *Management Science*, **38** (1992), 1732–1749.
[12] G. Erickson, *Dynamic Models of Advertising Competition*, 2nd edition, Kluwer, Norwell, MA, 2003.
[13] G. Erickson, An oligopoly model of dynamic advertising competition, *European Journal of Operational Research*, **197** (2009), 374–388.
[14] G. Feichtinger, R. F. l. Hart and S. P. Sethi, Dynamic optimal control models in advertising: Recent developments, *Management Science*, **40** (1994), 195–226.
[15] G. Fruchter, The many-player advertising game, *Management Science*, **45** (1999), 1609–1611.
[16] G. Fruchter and S. Kalish, Closed-loop advertising strategies in a duopoly, *Management Science*, **43** (1997), 54–63.
[17] E. González and M. Villena, Spatial lanchester models, *European Journal of Operational Research*, **210** (2011), 706–715.
[18] M. L. Huang J. and L. Liang, Recent developments in dynamic advertising research, *European Journal of Operational Research*, **220** (2012), 591–609.
[19] R. S. Jarmin, K. Shawn and J. Miranda, The role of retail chains: National, regional, and industry results, in *Producer Dynamics: New Evidence from Micro Data*, University of Chicago Press, 2009, 237–262.
[20] S. Jørgensen, A survey of some differential games in advertising, *Journal of Economic Dynamics and Control*, **4** (1982), 341–369.
[21] S. Jørgensen and G. Zaccour, *Differential Games in Marketing*, vol. 15, Springer, 2004.
[22] G. Kimball, Some industrial applications of military operations research methods, *Operations Research*, **5** (1957), 201–204.
[23] A. Krishnamoorthy, A. Prasad and S. Sethi, Optimal pricing and advertising in a durable-good duopoly, *European Journal of Operational Research*, **200** (2010), 486–497.
[24] F. W. Lanchester, *Aircraft in Warfare: The Dawn of the Fourth Arm*, Constable limited, 1916.
[25] J. D. Little, Aggregate advertising models: The state of the art, *Operations research*, **27** (1979), 629–667.
[26] C. Marinelli and S. Savin, Optimal distributed dynamic advertising, *Journal of Optimization Theory and Applications*, **137** (2008), 569–591.
[27] M. Nerlove and K. Arrow, Optimal advertising policy under dynamic conditions, *Mathematical Models in Marketing*, **132** (1976), 129–142.
[28] A. Prasad and S. P. Sethi, Advertising under uncertainty: A stochastic differential game approach, *Journal of Optimization Theory and Applications*, **123** (2004), 163–185.

[29] S. P. Sethi, Deterministic and stochastic optimization of a dynamic advertising model, *Optimal Control Applications and Methods*, **4** (1983), 179–184.

[30] S. P. Sethi and G. L. Thompson, *Optimal Control Theory: Applications to Management Science and Economics*, Kluwer, Norwell, MA, 2000.

[31] G. Sorger, Competitive dynamic advertising: A modification of the case game, *Journal of Economics Dynamics and Control*, **13** (1989), 55–80.

[32] M. Vidale and H. Wolfe, An operations research study of sales response to advertising, *Operations Research*, **5** (1957), 370–381.

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